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Low-energy absorption cross section for massive scalar and Dirac fermion by $(4+n)$-dimensional Schwarzschild black hole

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Abstract: Motivated by the brane-world scenarios, we study the absorption problem when the spacetime background is $(4+n)$-dimensional Schwarzschild black hole. We compute the low-energy absorption cross sections for the brane-localized massive scalar, brane-localized massive Dirac fermion, and massive bulk scalar. For the case of brane-localized massive Dirac fermion we introduce the particle's spin in the traditional Dirac form without invoking the Newman-Penrose method. Our direct introduction of spin enables us to compute contributions to the jth-level partial absorption cross section from orbital angular momenta $\ell = j \pm 1/2$. It is shown that the contribution from the low ℓ-level is larger than that from the high ℓ-level in the massive case. In the massless case these two contributions are exactly same with each other. The ratio of low-energy absorption cross sections for Dirac fermion and for scalar is dependent on the number of extra dimensions as $2^{-(n+3)/(n+1)}$. Thus the ratio factor $1/8$ is recovered when $n = 0$, which Unruh found. The physical importance of this ratio factor is discussed in the context of the brane-world scenario. For the case of bulk scalar our low-energy absorption cross section for S-wave is exactly same with area of the horizon hypersurface in the massless limt, which is an higher-dimensional generaliztion of universality. Our results for all cases turn out to have correct massless and $4d$ limits.

Keywords: Extra Large Dimensions, Black Holes.

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Contents

1. Introduction

The "greybody factor" $\Gamma_\ell(\omega)$ is an important quantity to understand the absorption and emission phenomena of a black hole. It is this factor which makes a black hole to be different from a black body. The physical origin of this factor is an effective potential barrier generated by a black hole spacetime. For example, the potential for the massless scalar generated by the $4d$ Schwarzschild spacetime is

$$V_{\text{eff}}(r_*) = \left(1 - \frac{r_H}{r}\right)\left(\frac{r_H}{r^3} + \frac{\ell(\ell+1)}{r^2}\right) \tag{1.1}$$

when the wave equation is expressed in terms of the "tortoise" coordinate $r_* = r + r_H \ln(r/r_H - 1)$, where r_H is an horizon radius and ℓ is an angular momentum of the scalar. This potential generally backscatters a part of the outgoing radiation quantum machanically, which results in the frequency-dependent greybody factor $\Gamma_\ell(\omega)$. This factor, therefore, appears in the black hole's thermal radiation formula

$$\frac{dH}{d\omega} = \sum_\ell \frac{\Gamma_\ell(\omega)}{e^{\omega/T_{BH}} - 1} \frac{(2\ell+1)\omega}{\pi} \tag{1.2}$$

where T_{BH} is an Hawking temperature. Eq. (1.2) shows how the greybody factor and the Planck factor play important roles in the emission rate of energy.

In addition to the radiation formula (1.2) the factor $\Gamma_\ell(\omega)$ is important to compute the partial absorption cross section $\sigma_\ell(\omega)$ of the black hole. The explicit relation[1] between $\sigma_\ell(\omega)$ and $\Gamma_\ell(\omega)$ for the $4d$ massive scalar is

$$\sigma_\ell(\omega) = \frac{\pi}{\omega^2 v^2}(2\ell+1)\Gamma_\ell(\omega) \tag{1.3}$$

and $v = \sqrt{1 - m^2/\omega^2}$ and m is particle's mass. Thus one can compute σ_ℓ from Γ_ℓ straightforwardly or *vice versa*.

[1] The relation between $\sigma_\ell(\omega)$ and $\Gamma_\ell(\omega)$ in higher dimensions is given in eq. (4.23).

Many computational techniques for calculation of the absorption cross section in the various $4d$ black hole were developed long ago [1]–[7]. The main streams of this procedure are to derive the solutions of the given wave equation in the near-horizon and asymptotic regimes separately and to match them in the appropriate intermediate place. Following this procedure Unruh [6] computed the low energy absorption cross section for the massive scalar and Dirac fermion in the $4d$ Schwarzschild background. It is instructive to introduce an explicit expression of ref. [6] for the massive scalar:

$$\begin{aligned}(\sigma_\ell)_{\rm unruh} &= \frac{\pi}{k^2 v^2}(2\ell+1)T_\ell \\ T_\ell &= \frac{\pi(\ell!)^4 2^{2\ell+2}(1+v^2)k^{2\ell+3}v^{2\ell}}{(2\ell!)^4(2\ell+1)^2[1-\exp\{-\pi k(1+v^2)/v\}]}\prod_{s=1}^{\ell}\left[s^2+\left(\frac{k(1+v^2)}{2v}\right)^2\right]. \end{aligned} \tag{1.4}$$

The most interesting one eq. (1.4) suggests is the fact that the low-energy absorption cross section for S-wave is equal to the horizon area in the massless limit. This is an universal property for the minimally coupled massless scalar. This universality indicates that the low-energy cross section encodes information on the near-horizon structure of black hole. Another interesting result of ref. [6] is the fact that the low-energy absorption cross section for Dirac fermion with mass μ is exactly $1/8$ of that for scalar with mass m if $\mu = m$. It is still unclear at least for us the physical origin of this ratio factor.

The universality is re-examined in the arbitrary dimensional spherically symmetric black hole [8]. Ref. [8] has shown that the low-energy absorption cross sections for scalar is equal to the horizon area while that for spin-1/2 particle is an area measured in a flat spatial metric conformally related to the true metric. The universality for the minimally coupled massless scalar is extended to the p-brane-like objects [9] and its generalization to the massive scalar is discussed recently [10, 11, 12]. In particular, it is shown in ref. [11] that the mass-dependence of the absorption cross section is very sensitive to the near-horizon structure of spacetime.

The computation of the low-energy absorption cross section is also important subject in the context of string theories and brane-world scenario. In string theories the black hole is effectively represented by a collective states [13] and the relevant low energy excitations of this effective description are the right- and left-moving modes of the string. The computation of the low-energy absorption cross section with this picture suggests that the effective string theory for the black hole is "heterotic", i.e. the right-moving sector has both fermionic and bosonic degrees of freedom, while the left-moving sector has only bosonic degree of freedom [14, 15].

In the context of the brane-world scenario the most remarkable fact is that the fundamental Planck mass can be low as a TeV scale. This TeV-scale gravity is realized by making use of the large extra dimensions [16, 17] or warped extra dimensions [18]. One of the striking consequences arising due to the TeV-scale gravity is that the future high-energy colliders such as the CERN Large Hadron Collider can be a black hole factory [19, 20, 21]. If the black holes can be really produced in the future collider, one can examine the quantum gravity effects of black hole in the laboratory such as Hawking radiation and/or information loss problem [22, 23]. Thus, it is important to investigate the absorption and emission

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problems in this context. Recently works along this direction were done [24, 25, 26]. In ref. [24] the low-energy absorption problems for massless bulk and brane-localized scalar are examined. In ref. [25] same problems for brane-localized massless particle with spin 1/2 and 1 are examined. The absorption and emission problems for the brane-localized particles in the full range of energy are numerically studied in ref. [26].

In this paper we would like to extend ref. [24, 25] by considering the massive spin-0 and spin-1/2 particles in the $(4+n)$-dimensional Schwarzschild background. Although the spin can be introduced in the curved spacetime with ease by making use of the Newman-Penrose formalism [27], we will introduce it in the more traditional Dirac form. In section 2 we will compute the low-energy absorption cross section for the brane-localized massive scalar. It is shown in this section that our final result for the low-energy absorption cross section has a correct massless limit. However, the $4d$ limit of it is not exactly same with eq. (1.4), but coincides with the low-energy expansion of eq. (1.4). The reason for this is explained in the appendix. In section 3 the low-energy absorption cross section for the massive Dirac fermion is computed by solving Dirac equation. The final form of the low-energy absorption cross section $\sigma_j(\omega)$, where j is a total angular momentum, is just sum of two contributions from orbital angular momenta $\ell = j \pm 1/2$, i.e. $\sigma_j(\omega) = \sigma_{j,\ell=j+1/2}(\omega) + \sigma_{j,\ell=j-1/2}(\omega)$. While the introduction of spin via usual Newman-Penrose formalism generally does not allow to compute each contribution, our introduction of spin enables us to compute $\sigma_{j,\ell=j+1/2}(\omega)$ and $\sigma_{j,\ell=j-1/2}(\omega)$ separately. It is shown in this section that the contribution from lower ℓ-state is larger than that from higher ℓ-state in the massive case. However, these two contributions to $\sigma_j(\omega)$ are exactly same in the massless limit. It is also shown that the ratio of the absorption cross section for massive scalar and massive Dirac fermion is $2^{-(n+3)/(n+1)}$, which gives a factor 1/8 when $n = 0$ as Unruh found. In section 4 the low-energy absorption cross section for bulk scalar is computed. It is shown that the massless and S-wave limit of our result for the absorption cross section is same with the area of the horizon hypersurface. Thus, the universality for S-wave is preserved in the higher-dimensional theories. In section 5 a brief conclusion is given.

2. Low-energy absorption cross section for brane-localized scalar

The various higher-dimensional black hole solutions of the Einstein equation were discussed in detail in ref. [28]. The explicit expression of the $(4+n)$-dimensional Schwarzschild solution in terms of the usual Schwarzschild coordinates is in the following:

$$ds^2 = -h(r)dt^2 + h(r)^{-1}dr^2 + r^2 d\Omega_{n+2}^2 \tag{2.1}$$

where

$$h(r) = 1 - \left(\frac{r_H}{r}\right)^{n+1} \tag{2.2}$$

and the angular part is given by

$$d\Omega_{n+2}^2 = d\theta_{n+1}^2 + \sin^2\theta_{n+1}\Big(d\theta_n^2 + \sin^2\theta_n\left(\cdots + \sin^2\theta_2\left(d\theta_1^2 + \sin^2\theta_1 d\phi^2\right)\cdots\right)\Big). \tag{2.3}$$

The horizon radius r_H is related to the black hole mass M as following:

$$r_H^{n+1} = \frac{8\Gamma\left(\frac{n+3}{2}\right)}{(n+2)\pi^{\frac{n+1}{2}}} \frac{M}{M_*^{n+2}} \tag{2.4}$$

where $M_* = G^{-1/(n+2)}$ is a $(4+n)$-dimensional Planck mass and G is a Newton constant.

Since we are interested in the scalar localized on the brane in this section, we assume the scalar field Φ is a function of only t, r, θ_1, and ϕ. Thus, letting $\Phi = e^{-i\omega t} R_{\omega\ell}(r) Y_\ell(\theta_1, \phi)$ yields a following radial equation

$$\frac{h(r)}{r^2} \frac{d}{dr}\left[h(r) r^2 \frac{dR}{dr}\right] + \left[\omega^2 - h(r)\left\{m^2 + \frac{\ell(\ell+1)}{r^2}\right\}\right] R = 0\,. \tag{2.5}$$

For the quantum-mechanical interpretation we can change eq. (2.5) into the following Schrödinger-like equation

$$-\frac{1}{2M_{\text{eff}}} \frac{d^2\psi_\ell}{dr_*^2} + V_{\text{eff}}(r_*)\psi_\ell = \omega^2 v^2 \psi_\ell \tag{2.6}$$

where

$$\begin{aligned} R &= r^{\frac{n}{2}} \psi_\ell\,, \qquad r_* = r_H \ln h \\ M_{\text{eff}}^{-1} &= 2(n+1)^2 (1-h)^{\frac{2n+4}{n+1}} \\ V_{\text{eff}}(r_*) &= h \frac{\ell(\ell+1) - \frac{n}{2}\left[(n+1) - \frac{n}{2}h\right]}{r^2} - (1-h)m^2\,. \end{aligned} \tag{2.7}$$

Unlike the usual $4d$ black hole case the Schrödinger-like equation (2.6) involves not only the effective potential but also the effective position-dependent mass. Thus, it seems to be difficult to get a direct quantum-mechanical interpretation from (2.6). However, one can roughly estimate the effect of extra dimension in the absorption cross section. For simplicity let us consider the massless limit. The effective potential as a function of r_* is plotted in figure 1 at fixed ℓ. Figure 1 indicates that the potential barrier height seems to increase with increasing the number of extra dimensions. Since the greybody factor usually decreases when potential barrier becomes higher and the absorption cross section is proportional to this factor, one can conjecture that the existence of the extra dimensions may reduces the absorption cross section, which explains figure 1 of ref. [26]. The detailed analysis of the Schrödinger-like equation (2.6) will be discussed elsewhere.

In ref. [6] Unruh derived the low-energy absorption cross section by solving the wave equation, i.e. the corresponding equation of eq. (2.5) in $4d$ Schwarzschild black hole, in the near-horizon, asymptotic and intermediate regions separately and matching the near-horizon and asymptotic solutions via the solution in the intermediate region. One may follow this procedure, but it seems to be impossible unlike $4d$ case to derive a solution in the intermediate region analytically. Thus, we adopt an alternative method introduced by Maldacena and Strominger in ref. [14]. As will be shown in the next section, however, this alternative method does not work when we discuss the absorption problem for Dirac fermion. In that case we will adopt the Unruh's original method.

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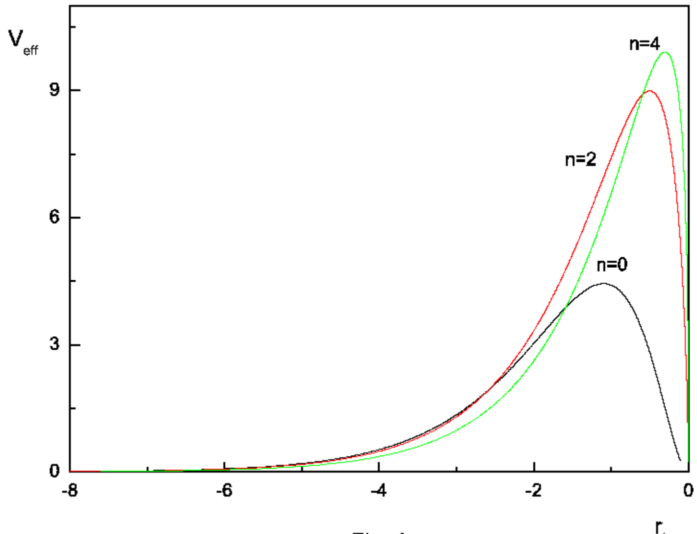

Fig. 1

Figure 1: Plot of effective potential V_{eff} given in eq. (2.7). We plotted the n-dependence of the effective potential at $\ell = 5$. Since the barrier height increases with increasing the number of extra dimensions, this figure indicates the existence of the extra dimensions may decrease the absorption cross section for the massive scalar.

Changing a variable makes eq. (2.5) to be

$$h(1-h)\frac{d^2R}{dh^2}+\left(1-\frac{2n+1}{n+1}h\right)\frac{dR}{dh}+\frac{1}{(n+1)^2}\left[\frac{(\omega r)^2}{h(1-h)}-\frac{(mr)^2+\ell(\ell+1)}{1-h}\right]R=0\,. \quad (2.8)$$

In the near-horizon region, i.e. $r \sim r_H$, eq. (2.8) is solved in terms of the hypergeometric function as follows:

$$\begin{aligned} R_{NH}(r) \,=\, & h^{\alpha}(1-h)^{\beta}\left[A_-F\left(\alpha+\beta+\frac{n}{1+n},\alpha+\beta;1+2\alpha;h\right)+\right. \\ & \left. +A_+h^{-2\alpha}F\left(-\alpha+\beta+\frac{n}{1+n},-\alpha+\beta;1-2\alpha;h\right)\right] \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} \alpha &= -\frac{i\omega r_H}{n+1} \\ \beta &= \frac{1}{2(n+1)}\left[1-\sqrt{(2\ell+1)^2-4\omega^2v^2r_H^2}\right]. \end{aligned} \quad (2.10)$$

Since $h \sim e^{r_*/r_H}$ in terms of tortoise coordinate introduced in eq. (2.7) and only outgoing wave is valid in the near-horizon region, we should impose $A_+ = 0$ and then $R_{NH}(r)$ reduces to

$$R_{NH}(r) \sim A_- e^{-i\frac{\omega r_*}{n+1}}\,. \quad (2.11)$$

Now, let us solve eq. (2.5) in the asymptotic region. Putting $h \sim 1$ makes eq. (2.5) in the following simple form;

$$\frac{d^2R_{FF}}{dr^2}+\frac{2}{r}\frac{dR_{FF}}{dr}+\left[\omega^2v^2-\frac{\ell(\ell+1)}{r^2}\right]R_{FF}=0 \quad (2.12)$$

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which yields an solution in terms of the usual Bessel function as following:

$$R_{FF} = \frac{1}{\sqrt{r}}\left[B_+ J_{\ell+\frac{1}{2}}(\omega v r) + B_- Y_{\ell+\frac{1}{2}}(\omega v r)\right]. \qquad (2.13)$$

To match the near-horizon solution $R_{NH}(r)$ in eq. (2.9) with $A_+ = 0$ and the far-field $R_{FF}(r)$ in eq. (2.13), we change the near-horizon solution as following:

$$\begin{aligned} R_{NH}(r) \;=\; & A_- h^{\alpha}(1-h)^{\beta}\left[\frac{\Gamma(1+2\alpha)\,\Gamma\left(-2\beta+\frac{1}{n+1}\right)}{\Gamma\left(\alpha-\beta+\frac{1}{n+1}\right)\Gamma(1+\alpha-\beta)}\times\right. \\ & \times F\left(\alpha+\beta+\frac{n}{n+1}, \alpha+\beta; 2\beta+\frac{n}{n+1}; 1-h\right) + \\ & + (1-h)^{-2\beta+\frac{1}{n+1}}\frac{\Gamma(1+2\alpha)\,\Gamma\left(2\beta-\frac{1}{n+1}\right)}{\Gamma\left(\alpha+\beta+\frac{n}{n+1}\right)\Gamma(\alpha+\beta)}\times \\ & \left.\times F\left(\alpha-\beta+\frac{1}{n+1}, 1+\alpha-\beta; -2\beta+\frac{n+2}{n+1}; 1-h\right)\right]. \end{aligned} \qquad (2.14)$$

Taking $R_{NH}(r)$ in eq. (2.14) to $r \to \infty$ and $R_{FF}(r)$ in eq. (2.13) to $r \to 0$ naturally yields two relations, one between A_- and B_+ and the other between A_- and B_-. Then removing A_- yields

$$B \equiv \frac{B_+}{B_-} = -\left(\frac{2}{\omega v r_H}\right)^{2\ell+1}\frac{\left(\ell+\frac{1}{2}\right)\Gamma^2\left(\ell+\frac{1}{2}\right)\Gamma\left(\frac{1}{n+1}-2\beta\right)\Gamma\left(\alpha+\beta+\frac{n}{n+1}\right)\Gamma(\alpha+\beta)}{\pi\Gamma(1+\alpha-\beta)\,\Gamma\left(\alpha-\beta+\frac{1}{n+1}\right)\Gamma\left(2\beta-\frac{1}{n+1}\right)}. \qquad (2.15)$$

Computing the flux in the asymptotic region, one can calculate the greybody factor whose expression is

$$\Gamma_\ell(\omega) = 1 - \left|\frac{B-i}{B+i}\right|^2 = \frac{2i(B^*-B)}{|B|^2 + i(B^*-B) + 1} \approx \frac{2i(B^*-B)}{|B|^2}. \qquad (2.16)$$

The last approximation in eq. (2.16) comes from the low-energy approximation, i.e. $\omega \ll 1$. Inserting eq. (2.15) into eq. (2.16) makes the greybody factor to be

$$\Gamma_\ell(\omega) = \frac{16\pi}{(n+1)^2 v}\left(\frac{\omega v r_H}{2}\right)^{2\ell+2}\frac{\Gamma^2\left(1+\frac{\ell}{n+1}\right)\Gamma^2\left(\frac{\ell+1}{n+1}\right)}{\Gamma^2\left(\ell+\frac{1}{2}\right)\Gamma^2\left(1+\frac{2\ell+1}{n+1}\right)}. \qquad (2.17)$$

Then the partial absorption cross section σ_ℓ can be read straightforwardly from eq. (2.17);

$$\sigma_\ell \equiv \frac{\pi(2\ell+1)}{\omega^2 v^2}\Gamma_\ell(\omega) = \frac{16\pi^2(2\ell+1)}{(n+1)^2\omega^2 v^3}\left(\frac{\omega v r_H}{2}\right)^{2\ell+2}\frac{\Gamma^2\left(1+\frac{\ell}{n+1}\right)\Gamma^2\left(\frac{\ell+1}{n+1}\right)}{\Gamma^2\left(\ell+\frac{1}{2}\right)\Gamma^2\left(1+\frac{2\ell+1}{n+1}\right)}. \qquad (2.18)$$

It is interesting to consider several limiting cases. Firstly, let us consider the massless limit, i.e. $v \approx 1$. Then the partial absorption cross section reduces to

$$\sigma_\ell^{m=0} = \frac{16\pi^2(2\ell+1)}{(n+1)^2\omega^2}\left(\frac{\omega r_H}{2}\right)^{2\ell+2}\frac{\Gamma^2\left(1+\frac{\ell}{n+1}\right)\Gamma^2\left(\frac{\ell+1}{n+1}\right)}{\Gamma^2\left(\ell+\frac{1}{2}\right)\Gamma^2\left(1+\frac{2\ell+1}{n+1}\right)} \qquad (2.19)$$

which is exactly same with the result of ref. [24]. In $n \to 0$ limit eq. (2.18) becomes

$$\sigma_\ell^{n=0} = \frac{\pi (\ell!)^6 2^{2\ell+2} \omega^{2\ell} v^{2\ell-1}}{(2\ell!)^4 (2\ell+1)} r_H^{2\ell+2} \tag{2.20}$$

which coincides with eq. (1.4) if we expands eq. (1.4) by making use of $\omega \to 0$ limit. Thus our result (2.18) recovers not only the massless limit but also $4d$ limit.

At this stage one may question why our $4d$ limit is not exactly eq. (1.4) but low-energy expansion of it. In fact, Unruh derived eq. (1.4) by expressing the asymptotic solution in terms of the Coulomb wave functions. The multiplicative factor and exponential factor in eq. (1.4) are results of these Coulomb wave functions. In appendix, however, we will show that the expression of asymptotic solution in terms of the Coulomb wave functions is impossible except $n = 0$. This is why the $4d$ limit of our result gives the low-energy expansion of eq. (1.4).

3. Low-energy absorption cross section for Dirac fermion

In this section we will compute the low-energy absorption cross section for the massive Dirac fermion in the Schwarzschild background defined on the bulk. As commented earlier, spin of the particle can be introduced more conveniently by exploiting the Newman-Penrose formalism. However, it is more straightforward to introduce it in the traditional Dirac form, which we will follow in this section. As will be shown shortly, furthermore, our introduction of spin enables us to compute each contribution to the low-energy partial absorption cross section. The usual Newman-Penrose formalism does not provide this information. It usually gives the total sum of it. Thus, we can determine which contribution is dominant.

Let us start with Dirac equation

$$[\gamma^\mu (\partial_\mu - \Gamma_\mu) + \mu]\, \psi = 0 \tag{3.1}$$

where μ is a mass of the fermion and Γ_μ is a spin-affine connection. Since we are interested in the Dirac fermion localized on the brane, we assume ψ is function of t, r, $\theta = \theta_1$ and ϕ. The gamma matrices in this background can be easily chosen as $\gamma^t = \gamma^0/\sqrt{h}$, $\gamma^r = \sqrt{h}\gamma^3$, $\gamma^\theta = \gamma^1/r$, and $\gamma^\phi = \gamma^2/r \sin\theta$ where γ^i are the ordinary flat-space Dirac matrices defined

$$\gamma^0 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{3.2}$$

and σ^i are usual Pauli matrices.

The spin-affine connection in the $(4+n)$-dimensional Schwarzschild background can be straightforwardly computed with use of the affine connections and the results are

$$\begin{aligned} \Gamma_t &= -\frac{(1+n)(1-h)}{4r}\gamma^3\gamma^0 & \Gamma_r &= 0 \\ \Gamma_\theta &= \frac{\sqrt{h}}{2}\gamma^3\gamma^1 & \Gamma_\phi &= \frac{\sqrt{h}}{2}\sin\theta\gamma^3\gamma^2 + \frac{1}{2}\cos\theta\gamma^1\gamma^2\,. \end{aligned} \tag{3.3}$$

Then, Dirac operator $\gamma^\mu(\partial_\mu - \Gamma_\mu) + \mu$ reduces to

$$\gamma^\mu(\partial_\mu - \Gamma_\mu) + \mu = \begin{pmatrix} \frac{1}{\sqrt{h}}\partial_t + i\mu & \hat{H} \\ -\hat{H} & -\frac{1}{\sqrt{h}}\partial_t + i\mu \end{pmatrix} \tag{3.4}$$

where

$$\hat{H} = \sqrt{h}\sigma^3 D_r + \frac{\sigma^1}{r} D_\theta + \frac{\sigma^2}{r\sin\theta}\partial_\phi \tag{3.5}$$

and

$$\begin{aligned} D_r &= \partial_r + \frac{1}{r} + \frac{(n+1)(1-h)}{4hr} \\ D_\theta &= \partial_\theta + \frac{\cot\theta}{2}. \end{aligned} \tag{3.6}$$

In order to perform the separation of variables we take an following ansatz

$$\psi = \frac{e^{-i\epsilon t}}{r h^{\frac{1}{4}}} \begin{pmatrix} G(r)\hat{\Theta}(\theta,\phi) \\ -iF(r)\sigma^3\hat{\Theta}(\theta,\phi) \end{pmatrix} \tag{3.7}$$

where $\hat{\Theta}(\theta,\phi)$ is angle-dependent two-component spinor. Then, it is straightforward to derive two radial equations

$$\begin{aligned} \sqrt{h}\,\frac{dG}{dr} + \frac{k}{r}G &= \left(\frac{\epsilon}{\sqrt{h}} + \mu\right) F \\ \sqrt{h}\,\frac{dF}{dr} - \frac{k}{r}F &= \left(\frac{-\epsilon}{\sqrt{h}} + \mu\right) G \end{aligned} \tag{3.8}$$

where k is defined from the angular equation

$$\left(\sigma^2 D_\theta - \frac{\sigma^1}{\sin\theta}\partial_\phi\right)\hat{\Theta} = ik\hat{\Theta}\,. \tag{3.9}$$

The angular equation was discussed long ago by Schrödinger [29] and the constant k is related to the orbital angular momentum ℓ by $\ell = |k+1/2| - 1/2$ and to the total angular momentum j by $j = |k| - 1/2$ [6]. Thus, for example, the lowest quantum number $j = 1/2$ corresponds to $k = -1$ ($\ell = 0$) and $k = 1$ ($\ell = 1$).

Removing the low component $F(r)$ in eq. (3.8), we can finally derive a second-order radial differential equation:

$$\frac{d^2G}{dx^2} + \left[\epsilon^2\left(\frac{1-\lambda\sqrt{h}}{1+\lambda\sqrt{h}}\right) - \frac{k^2 h}{(1+\lambda\sqrt{h})^2 r^2} + \frac{d}{dx}\left(\frac{k\sqrt{h}}{(1+\lambda\sqrt{h})r}\right)\right] G = 0 \tag{3.10}$$

where $\lambda = \mu/\epsilon$ and x is defined as

$$\frac{dx}{dr} = \frac{1+\lambda\sqrt{h}}{h}\,. \tag{3.11}$$

It is important to note that the new variable x goes to $x \sim (r_H/(n+1))\ln h$ in the near-horizon region and $x \sim (1+\lambda)r$ in the asymptotic region.

Due to the $\sqrt{h}$ in eq. (3.10), the method of Maldacena and Strominger in ref. [14] does not work in this case. Thus, we will adopt the method Unruh did in ref. [6].

Eq. (3.10) indicates that the radial equation in the near-horizon becomes simply

$$\frac{d^2 G_{NH}}{dx^2} + \epsilon^2 G_{NH} = 0 \tag{3.12}$$

which makes the following outgoing wave

$$G_{NH} = \alpha_I e^{-i\epsilon x} \approx \alpha_I e^{-i\frac{\epsilon r_H}{n+1}\ln h}\,. \tag{3.13}$$

In the intermediate region where ϵ^2 and μ^2 are much smaller than the other terms the radial equation (3.10) reduces to

$$\frac{d^2 G_{IM}}{dx^2} + \left[\frac{-k^2 h}{(1+\lambda\sqrt{h})^2 r^2} + \frac{d}{dx}\left(\frac{k\sqrt{h}}{(1+\lambda\sqrt{h})r}\right)\right] G_{IM} = 0\,. \tag{3.14}$$

Defining

$$H_{IM} \equiv \frac{dG_{IM}}{dx} + \frac{k\sqrt{h}}{(1+\lambda\sqrt{h})r} G_{IM}\,, \tag{3.15}$$

we can change eq. (3.14) into the first-order differential equation in the form

$$\frac{dH_{IM}}{dr} - \frac{k}{\sqrt{h}r} H_{IM} = 0\,. \tag{3.16}$$

The solution of (3.16) is

$$H_{IM} = \beta_{II}\left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right)^{-\frac{k}{n+1}}\,. \tag{3.17}$$

Therefore, inserting eq. (3.17) into eq. (3.15), we can obtain the following G_{IM}

$$G_{IM} = \alpha_{II}\left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right)^{\frac{k}{n+1}} + \beta_{II}\mathcal{G_{IM}} \tag{3.18}$$

where $\mathcal{G_{IM}}$ is a particular solution to

$$\frac{d\mathcal{G_{IM}}}{dr} + \frac{k}{\sqrt{h}r}\mathcal{G_{IM}} = \frac{1+\lambda\sqrt{h}}{h}\left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right)^{-\frac{k}{n+1}}\,. \tag{3.19}$$

For $k<0$ we obtain

$$\mathcal{G_{IM}} = \left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right)^{\frac{k}{n+1}} \int_1^{\sqrt{h}} \frac{2(1+\lambda\rho)}{(n+1)\rho}\frac{(1-\rho)^{\frac{2|k|-(n+2)}{n+1}}}{(1+\rho)^{\frac{2|k|+(n+2)}{n+1}}} d\rho\,, \tag{3.20}$$

whereas for $k>0$ we can show

$$\mathcal{G_{IM}} = \left(\frac{1-\sqrt{h}}{1+\sqrt{h}}\right)^{\frac{k}{n+1}} \left\{\frac{1}{n+1}\ln h + \int_0^{\sqrt{h}} d\rho \frac{2}{(n+1)\rho}\left[\frac{(1+\lambda\rho)}{(1-\rho^2)^{\frac{n+2}{n+1}}}\left(\frac{1+\rho}{1-\rho}\right)^{\frac{2k}{n+1}} - 1\right]\right\}. \tag{3.21}$$

In the asymptotic region eq. (3.10) reduces to

$$\frac{d^2 G_{FF}}{dx^2} + \left[\epsilon^2 \frac{1-\lambda}{1+\lambda} - \frac{k(k+1)}{x^2}\right] G_{FF} = 0\,. \tag{3.22}$$

Since $x \sim (1+\lambda)r$ in the asymptotic region, the solution of eq. (3.22) can be written as

$$G_{FF}(r) = \alpha_{III}\sqrt{\frac{r}{r_H}}\, J_{|k+\frac{1}{2}|}(\epsilon v r) + \beta_{III}\sqrt{\frac{r}{r_H}}\, Y_{|k+\frac{1}{2}|}(\epsilon v r) \tag{3.23}$$

where $v = \sqrt{1-\mu^2/\epsilon^2}$.

Now, let us consider a matching between G_{NH} and G_{IM}. Firstly, we note that the near-horizon solution (3.13) can be expanded as

$$G_{NH} \sim \alpha_I \left(1 - i\frac{\epsilon r_H}{n+1}\ln h + \cdots\right). \tag{3.24}$$

Secondly, let us consider $r \to r_H$ limit of G_{IM}. Eq. (3.19) indicates the $r \to r_H$ limit of G_{IM} becomes

$$\lim_{r\to r_H} G_{IM} \approx \alpha_{II} + \beta_{II} \lim_{r\to r_H} \mathcal{G_{IM}}\,. \tag{3.25}$$

For $k < 0$ eq. (3.20) implies

$$\lim_{r\to r_H} \mathcal{G_{IM}} \sim b_n + \frac{1}{n+1}\ln h \tag{3.26}$$

where b_n is a n-dependent finite quantity defined

$$b_n \sim \int_0^1 d\rho \frac{2}{(n+1)\rho}\left[1 - (1+\lambda\rho)\frac{(1-\rho)^{\frac{2|k|-(n+2)}{n+1}}}{(1+\rho)^{\frac{2|k|+(n+2)}{n+1}}}\right]. \tag{3.27}$$

Note that the factor 1 in the bracket compensates an infinity arising due to $1/\rho$ at $\rho \sim 0$. Thus, comparing eq. (3.25) with eq. (3.24) simply yield the matching conditions between G_{NH} and G_{IM}

$$\alpha_{II} = \alpha_I \qquad \beta_{II} = -i\epsilon r_H \alpha_I\,. \tag{3.28}$$

For $k > 0$, eq. (3.21) implies

$$\lim_{r\to r_H} G_{IM} \sim \alpha_{II} + \frac{\beta_{II}}{n+1}\ln h \tag{3.29}$$

which also gives eq. (3.28).

Next, let us consider a matching between G_{IM} and G_{FF}. Firstly, let us consider $k > 0$ case. Taking $r \to \infty$ limit to G_{IM} in eq. (3.18) and direct integration in eq. (3.21) yields

$$\begin{aligned}
\lim_{r\to\infty} G_{IM} &\sim \alpha_{II} 4^{-\frac{k}{n+1}}\left(\frac{r_H}{r}\right)^k + \beta_{II} 4^{-\frac{k}{n+1}}\left(\frac{r_H}{r}\right)^k \left[\mathcal{E} + \frac{1+\lambda}{2k+1} 4^{\frac{2k}{n+1}}\left(\frac{r}{r_H}\right)^{2k+1}\right] \\
&\sim \alpha_{II} 4^{-\frac{k}{n+1}}\left(\frac{r_H}{r}\right)^k + \beta_{II}\frac{(1+\lambda)4^{\frac{k}{n+1}}}{2k+1}\left(\frac{r}{r_H}\right)^{k+1}
\end{aligned} \tag{3.30}$$

where $\mathcal{E}$ is an integration constant assumed small with respect to r^{2k+1}.

Now, we expand $G_{FF}(r)$ in eq. (3.23) with assuming $\epsilon v r \ll 1$, which is valid in the low-energy approximation. Then, the usual Bessel function properties yield

$$\lim_{\epsilon v r \to 0} G_{FF}(r) \sim \frac{\alpha_{III}}{\sqrt{r_H}} \frac{\left(\frac{\epsilon v}{2}\right)^{k+\frac{1}{2}}}{\Gamma\left(k+\frac{3}{2}\right)} r^{k+1} - \frac{\beta_{III}}{\pi\sqrt{r_H}} \Gamma\left(k+\frac{1}{2}\right)\left(\frac{\epsilon v}{2}\right)^{-\left(k+\frac{1}{2}\right)} r^{-k}. \tag{3.31}$$

Comparision of eq. (3.31) with eq. (3.30), therefore, makes the following matching conditions

$$\begin{aligned} \alpha_{II} &= -\frac{2^{\frac{2k}{n+1}}\Gamma\left(k+\frac{1}{2}\right)}{\pi}\left(\frac{\epsilon v r_H}{2}\right)^{-\left(k+\frac{1}{2}\right)}\beta_{III} \\ \beta_{II} &= \frac{2^{1-\frac{2k}{n+1}}}{(1+\lambda)\Gamma\left(k+\frac{1}{2}\right)}\left(\frac{\epsilon v r_H}{2}\right)^{k+\frac{1}{2}}\alpha_{III}\,. \end{aligned} \tag{3.32}$$

A similar procedure leads the following matching conditions for $k<0$;

$$\begin{aligned} \alpha_{II} &= \frac{2^{\frac{2k}{n+1}}}{\Gamma\left(\frac{1}{2}-k\right)}\left(\frac{\epsilon v r_H}{2}\right)^{-\left(k+\frac{1}{2}\right)}\alpha_{III} \\ \beta_{II} &= \frac{2^{1-\frac{2k}{n+1}}\Gamma\left(-k+\frac{1}{2}\right)}{\pi(1+\lambda)}\left(\frac{\epsilon v r_H}{2}\right)^{k+\frac{1}{2}}\beta_{III}\,. \end{aligned} \tag{3.33}$$

Thus, eq. (3.28) and eq. (3.32) gives for $k>0$

$$\frac{\beta_{III}}{\alpha_{III}} = \frac{\pi v 2^{-\frac{4k}{n+1}}}{i(1+\lambda)\Gamma^2\left(k+\frac{1}{2}\right)}\left(\frac{\epsilon v r_H}{2}\right)^{2k} \tag{3.34}$$

and eq. (3.28) and eq. (3.33) yield for $k<0$

$$\frac{\beta_{III}}{\alpha_{III}} = \frac{-i\pi(1+\lambda)}{v 2^{-\frac{4k}{n+1}}\Gamma^2\left(\frac{1}{2}-k\right)}\left(\frac{\epsilon v r_H}{2}\right)^{-2k}. \tag{3.35}$$

Now, let us compute the low-energy absorption cross section. Computing the flux in the asymptotic region, one can easily express the greybody factor for Dirac fermion as

$$\Gamma_j(\epsilon) = \frac{2i\left(\frac{\beta_{III}}{\alpha_{III}} - \frac{\beta^*_{III}}{\alpha^*_{III}}\right)}{\left|1+i\frac{\beta_{III}}{\alpha_{III}}\right|^2}. \tag{3.36}$$

Thus for $k>0$ $\Gamma_j(\epsilon)$ reduces to

$$\Gamma_{j,k>0}(\epsilon) = \frac{\left(\frac{4\pi v 2^{-\frac{4k}{n+1}}\left(\frac{\epsilon v r_H}{2}\right)^{2k}}{(1+\lambda)\Gamma^2\left(k+\frac{1}{2}\right)}\right)}{\left[1+\frac{\pi v 2^{-\frac{4k}{n+1}}\left(\frac{\epsilon v r_H}{2}\right)^{2k}}{(1+\lambda)\Gamma^2\left(k+\frac{1}{2}\right)}\right]^2} \approx \frac{4\pi v 2^{-\frac{4k}{n+1}}\left(\frac{\epsilon v r_H}{2}\right)^{2k}}{(1+\lambda)\Gamma^2\left(k+\frac{1}{2}\right)} \tag{3.37}$$

and for $k < 0$

$$\Gamma_{j,k<0}(\epsilon) = \frac{\left(\frac{4\pi(1+\lambda)}{v2^{-\frac{4k}{n+1}}\Gamma^2\left(\frac{1}{2}-k\right)\left(\frac{\epsilon v r_H}{2}\right)^{2k}}\right)}{\left[1+\frac{\pi(1+\lambda)}{v2^{-\frac{4k}{n+1}}\Gamma^2\left(\frac{1}{2}-k\right)\left(\frac{\epsilon v r_H}{2}\right)^{2k}}\right]^2} \approx \frac{4\pi(1+\lambda)\left(\frac{\epsilon v r_H}{2}\right)^{-2k}}{v2^{-\frac{4k}{n+1}}\Gamma^2\left(\frac{1}{2}-k\right)} . \tag{3.38}$$

At this stage it is worthwhile noting the following. If one introduces a spin by Newman-Penrose formalism, only the greybody factor for each j-level can be calculated, which is just sum of $\Gamma_{j,k>0}(\epsilon)$ and $\Gamma_{j,k<0}(\epsilon)$. Since, however, we introduced a spin by usual Dirac form, we are able to calculate $\Gamma_{j,k>0}(\epsilon)$ and $\Gamma_{j,k<0}(\epsilon)$ individually, where the former corresponds to $\ell = j + 1/2$ and the latter to $\ell = j - 1/2$. Thus we can determine which contribution of angular momentum quantum number is dominant. If, for example, j is fixed, there are contribution to the greybody factor from $\ell = j + 1/2$ (or $k = \ell$) and $\ell = j - 1/2$ (or $k = -\ell - 1$). Then eq. (3.37) and (3.38) indicate

$$\frac{\Gamma_{j,k=j+\frac{1}{2}}}{\Gamma_{j,k=-j-\frac{1}{2}}} = \frac{1-\lambda}{1+\lambda} \leq 1 \tag{3.39}$$

Thus the contribution from $\ell = j - 1/2$ to the greybody factor is larger than that from $\ell = j + 1/2$. However, in the massless limit two contributions are exactly same.

The absorption cross section for fixed j is given by

$$\begin{aligned}\sigma_j(\epsilon) &= \frac{\pi(2j+1)}{2\epsilon^2 v^2}\left(\Gamma_{k=j+\frac{1}{2}}(\epsilon) + \Gamma_{k=-j-\frac{1}{2}}(\epsilon)\right) \\ &= \frac{\pi(2j+1)}{\epsilon^2 v^3}\frac{(2\pi)2^{-\frac{4j+2}{n+1}}}{2^{2j}\Gamma^2(j+1)}(\epsilon v r_H)^{2j+1}.\end{aligned} \tag{3.40}$$

Now, let us consider the limiting cases. In the massless limit eq. (3.40) shows

$$\sigma_j^{m=0}(\epsilon) = \frac{\pi(2j+1)}{\epsilon^2}\frac{(2\pi)2^{-\frac{4j+2}{n+1}}}{2^{2j}\Gamma^2(j+1)}(\epsilon r_H)^{2j+1} \tag{3.41}$$

which exactly coincides with eq. (45) of ref. [25].

Next, let us consider the case of the lowest angular momentum quantum number, i.e. $j = 1/2$. In this case the absorption cross section becomes

$$\sigma_{j=\frac{1}{2}}(\epsilon) = \frac{\pi}{v}2^{\frac{n-1}{n+1}}r_H^2 . \tag{3.42}$$

Since eq. (2.19) indicates the low-energy absorption cross section for S-wave is

$$\sigma_{\ell=0} = \frac{4\pi r_H^2}{v} , \tag{3.43}$$

the ratio between $\sigma_{j=1/2}$ and $\sigma_{\ell=0}$ when $\epsilon = \omega$ and $\mu = m$ becomes

$$\frac{\sigma_{j=\frac{1}{2}}}{\sigma_{\ell=0}} = 2^{-\frac{n+3}{n+1}} . \tag{3.44}$$

Thus we get $\sigma_{j=1/2}/\sigma_{\ell=0} = 1/8$ when $n = 0$, which Unruh found in his seminar paper in ref. [6]. Thus, our result eq. (3.44) is a generalized ratio between the low-energy absorption cross sections for spin-1/2 and scalar particles in the higher-dimensional brane-world theories. In conclusion, our result (3.40) correctly reproduces the $n = 0$ limit as well as $4d$ limit.

4. Low-energy absorption cross section for bulk scalar

In this section we will discuss on the low-energy absorption problem for the minimally-coupled massive scalar which lives in the bulk. Thus, we should assume the scalar field Φ is function of all bulk coordimates. Then, it is straightforward to show that the usual Klein-Gordon equation $(\Box - m^2)\Phi = 0$ in the $(4+n)$-dimensional Schwarzschild background (2.1) reduces to

$$\frac{h(r)}{r^{n+2}}\frac{d}{dr}\left[h(r)r^{n+2}\frac{dR}{dr}\right] + \left[\omega^2 - h(r)\left\{m^2 + \frac{\ell(\ell+n+1)}{r^2}\right\}\right]R = 0 \tag{4.1}$$

where we used $\Phi = e^{-i\omega t}R_{\omega\ell}(r)\tilde{Y}_\ell(\Omega)$ and $\tilde{Y}_\ell(\Omega)$ is an higher-dimensional spherical harmonics.

At this stage it is worthwhile noting that one can calculate the low-energy absorption cross section by using the method used in ref. [14] and Unruh's original method used in ref. [6]. Since both procedures yield a same result, we will adopt the latter in this paper.

In order to obtain the near-horizon solution of eq. (4.1) it is convenient to introduce a tortoise coordinate

$$y = \frac{1}{(n+1)r_H^{n+1}}\ln h(r) \tag{4.2}$$

which changes eq. (4.1) into

$$\frac{d^2R}{dy^2} + r^{2n+4}\left[\omega^2 - h(r)\left\{m^2 + \frac{\ell(\ell+n+1)}{r^2}\right\}\right]R = 0\,. \tag{4.3}$$

Since $h \sim 0$ in the near-horizon region, eq. (4.3) approximately reduces to

$$\frac{d^2R_{NH}}{dy^2} + r_H^{2n+4}\omega^2 R_{NH} \approx 0 \tag{4.4}$$

which gives the out-going wave

$$R_{NH} = A_I e^{-i\omega r_H^{n+2} y}\,. \tag{4.5}$$

In the intermediate region we use an inequality $h(r)\ell(\ell+n+1)/r^2 \gg \omega^2$ which makes eq. (4.1) to be in this region

$$\frac{d}{dr}\left[h(r)r^{n+2}\frac{dR_{IM}}{dr}\right] - r^n\ell(\ell+n+1)R_{IM} = 0\,. \tag{4.6}$$

It is easy to show that eq. (4.6) provides a solution in terms of the Legendre polynomials as following

$$R_{IM} = A_{II}P_{\frac{\ell}{n+1}}\left(2\left(\frac{r}{r_H}\right)^{n+1} - 1\right) + B_{II}Q_{\frac{\ell}{n+1}}\left(2\left(\frac{r}{r_H}\right)^{n+1} - 1\right). \tag{4.7}$$

In the asymptotic region $h(r) \sim 1$ and the radial equation (4.1) becomes approximately

$$\frac{1}{r^{n+2}}\frac{d}{dr}r^{n+2}\frac{dR_{FF}}{dr} + \left[(\omega^2 - m^2) - \frac{\ell(\ell+n+1)}{r^2}\right] R_{FF} = 0\,. \tag{4.8}$$

It is easy to show that eq. (4.8) is solved by

$$R_{FF} = \frac{1}{r^{\frac{n+1}{2}}}\left[A_{III} J_{\ell+\frac{n+1}{2}}(\omega v r) + B_{III} Y_{\ell+\frac{n+1}{2}}(\omega v r)\right] \tag{4.9}$$

where $v = \sqrt{1 - m^2/\omega^2}$.

Now, let us consider a matching between R_{NH} and R_{IM}. Using a relations between the Legendre polynomials and the hypergeometric finction

$$\begin{aligned} P_\nu(z) &= F\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right) \\ Q_\nu(z) &= 2^{-\nu-1}\sqrt{\pi}\frac{\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{3}{2}\right)} z^{-\nu-1} F\left(1+\frac{\nu}{2}, \frac{1}{2}+\frac{\nu}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right), \end{aligned} \tag{4.10}$$

the $r \to r_H$ limit of R_{IM} reduces to

$$\begin{aligned} \lim_{r\to r_H} R_{IM} &= A_{II}\left[1 + \ell\left(\frac{\ell}{n+1}+1\right)\left(\frac{r}{r_H}-1\right) + \cdots\right] \\ &\quad + \frac{B_{II}}{2}\left[-\ln\left(\frac{r}{r_H}-1\right) + \left\{2\psi(1) - \psi\left(1+\frac{\ell}{2n+2}\right) - \psi\left(\frac{1}{2}+\frac{\ell}{2n+2}\right) - \right.\right. \\ &\qquad\qquad \left.\left. - \ln 4(n+1)\right\} + \cdots\right] \end{aligned} \tag{4.11}$$

where $\psi(z)$ is an usual digamma function. Comparing eq. (4.11) with

$$R_{NH} \sim A_I\left[1 - \frac{i r_H \omega}{n+1}\left\{\ln\left(\frac{r}{r_H}-1\right) + \ln(n+1)\right\} + \cdots\right], \tag{4.12}$$

we can derive the matching conditions

$$\begin{aligned} A_{II} &= A_I\left[1 + \frac{i r_H \omega}{n+1}\left\{\psi\left(1+\frac{\ell}{2n+2}\right) + \psi\left(\frac{1}{2}+\frac{\ell}{2n+2}\right) + \gamma - \psi\left(\frac{1}{2}\right)\right\}\right] \approx A_I \\ B_{II} &= \frac{2 i r_H \omega}{n+1} A_I \end{aligned} \tag{4.13}$$

where γ is an Euler's constant and the last approximation in A_{II} comes from the low-energy approximation.

Next the matching between R_{IM} and R_{FF} will be discussed. Using relations (4.10) it is not difficult to show that the $r \to \infty$ limit of R_{IM} is

$$\lim_{r\to\infty} R_{IM} = A_{II}\frac{\Gamma\left(1+\frac{2\ell}{n+1}\right)}{\Gamma^2\left(1+\frac{\ell}{n+1}\right)}\left(\frac{r}{r_H}\right)^{\ell} + B_{II}\frac{\Gamma^2\left(1+\frac{\ell}{n+1}\right)}{2\Gamma\left(2+\frac{2\ell}{n+1}\right)}\left(\frac{r_H}{r}\right)^{n+\ell+1}. \tag{4.14}$$

Since $\omega v r \to 0$ limit of R_{FF} is

$$\lim_{\omega v r \to 0} R_{FF} = A_{III} \frac{\left(\frac{\omega v}{2}\right)^{\ell + \frac{n+1}{2}}}{\Gamma\left(\ell + \frac{n+3}{2}\right)} r^{\ell} - \frac{B_{III}}{\pi} \Gamma\left(\ell + \frac{n+1}{2}\right) \left(\frac{2}{\omega v}\right)^{\ell + \frac{n+1}{2}} r^{-\ell-n-1}, \tag{4.15}$$

the comparision of eq. (4.14) with eq. (4.15) gives the following natching conditions;

$$\begin{aligned} A_{III} &= \frac{\Gamma\left(1 + \frac{2\ell}{n+1}\right) \Gamma\left(\ell + \frac{n+3}{2}\right)}{r_H^{\ell} \Gamma^2\left(1 + \frac{\ell}{n+1}\right)} \left(\frac{\omega v}{2}\right)^{-\left(\ell + \frac{n+1}{2}\right)} A_{II} \\ B_{III} &= -\frac{\pi r_H^{n+\ell+1} \Gamma^2\left(1 + \frac{\ell}{n+1}\right)}{2\Gamma\left(2 + \frac{2\ell}{n+1}\right) \Gamma\left(\ell + \frac{n+1}{2}\right)} \left(\frac{\omega v}{2}\right)^{\ell + \frac{n+1}{2}} B_{II}. \end{aligned} \tag{4.16}$$

One should note that the expansion (4.15) of R_{FF} is valid in the low-energy limit.

Now, it is straightforward to calculate the low-energy absorption cross section for the bulk scalar. For the computation we first separate R_{FF} as a combination of incident and reflected waves;

$$\lim_{r \to \infty} R_{FF} = \phi_{\rm in} + \phi_{\rm re} \tag{4.17}$$

where

$$\begin{aligned} \phi_{\rm in} &= e^{i\frac{\pi}{2}\left(\ell + \frac{n}{2} + 1\right)} \sqrt{\frac{1}{2\pi\omega v}} (A_{III} + iB_{III}) \frac{e^{-i\omega v r}}{r^{\frac{n}{2}+1}} \\ \phi_{\rm re} &= e^{-i\frac{\pi}{2}\left(\ell + \frac{n}{2} + 1\right)} \sqrt{\frac{1}{2\pi\omega v}} (A_{III} - iB_{III}) \frac{e^{i\omega v r}}{r^{\frac{n}{2}+1}}. \end{aligned} \tag{4.18}$$

Then, the greybody factor is directly computed from the reflection coefficient

$$\Gamma_{\ell}(\omega) = 1 - \left| \frac{1 - i\frac{B_{III}}{A_{III}}}{1 + i\frac{B_{III}}{A_{III}}} \right|^2. \tag{4.19}$$

The matching conditions (4.13) and (4.16) gives

$$\frac{B_{III}}{A_{III}} = -iD \tag{4.20}$$

where

$$D = \frac{\pi^2}{2^{\frac{4\ell}{n+1}} v} \left(\frac{\omega v r_H}{2}\right)^{n+2\ell+2} \frac{\Gamma^2\left(1 + \frac{\ell}{n+1}\right)}{\Gamma^2\left(\frac{1}{2} + \frac{\ell}{n+1}\right) \Gamma^2\left(\frac{n+3}{2} + \ell\right)}. \tag{4.21}$$

Thus, the factor $\Gamma_{\ell}(\omega)$ reduces to

$$\Gamma_{\ell}(\omega) = \frac{4D}{(1+D)^2} \approx 4D = \frac{4\pi^2}{2^{\frac{4\ell}{n+1}} v} \left(\frac{\omega v r_H}{2}\right)^{n+2\ell+2} \frac{\Gamma^2\left(1 + \frac{\ell}{n+1}\right)}{\Gamma^2\left(\frac{1}{2} + \frac{\ell}{n+1}\right) \Gamma^2\left(\frac{n+3}{2} + \ell\right)}. \tag{4.22}$$

The massless limit of eq. (4.22) exactly coincides with eq. (37) of ref. [24].

The relation between the absorption cross section and the greybody factor can be derived using the $(4+n)$-dimensional optical theorem [30]

$$\sigma_\ell(\omega) = \frac{2^n \Gamma^2\left(\frac{n+3}{2}\right)}{\pi(\omega v r_H)^{n+2}} \tilde{A}_H \frac{(2\ell+n+1)(\ell+n)!}{(n+1)!\ell!} \Gamma_\ell(\omega) \tag{4.23}$$

where $\tilde{A}_H$ is an area of the horizon hypersurface

$$\tilde{A}_H = \frac{2\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} r_H^{n+2} . \tag{4.24}$$

The most important quantity is a low-energy absorption cross section for S-wave, which is derived by letting $\ell = 0$ in eq. (4.23)

$$\sigma_{\ell=0}(\omega) = \frac{\tilde{A}_H}{v} . \tag{4.25}$$

Thus, at the massless limit the absorption cross section for S-wave exactly coincides with the area of the horizon hypersurface $\tilde{A}_H$, which is an higher dimensional generalization of the universality. The $4d$ limit of σ_ℓ is easily computed by letting $n = 0$, which is exactly same with eq. (2.20). Also the massless limit of eq. (4.23) exactly coincides with eq. (5.4) of ref. [26]. Thus our result has correct $4d$ and massless limits.

5. Conclusion

The low-energy absorption cross section for the brane-localized massive scalar, brane-localized massive Dirac fermion, and massive bulk scalar are explicitly computed when the background is $(4+n)$-dimensional Schwarzschild spacetime.

For the case of the brane-localized massive scalar our result (2.18) has a correct massless limit. But the $4d$ limit of eq. (2.18) is not exactly same with the Unruh's $4d$ result (1.4), but coincides with its low-energy expansion. The reason for this is clarified in the appendix.

For the case of the brane-localized Dirac fermion we introduced the particle's spin using the traditional Dirac form instead of the Newman-Penrose formalism. Thus, we should compute the spin-affine connections and separate the Dirac equation to derive a radial equation, which were performed explicitly in section 3. Our introduction of spin enables us to compute the contributions to the low-energy absorption cross section from the orbital angular momentum quantum numbers ℓ. Since, for spin-1/2 particle, total angular momentum quantum number j is contributed from $\ell = j \pm 1/2$, the ratio of $\sigma_{j,\ell=j+1/2}$ and $\sigma_{j,\ell=j-1/2}$ is given in eq. (3.39). This equation indicates that the contribution from lower orbital angular momentum is larger than that from higher orbital angular momentum in the massive case. However, two contributions are exactly same in the massless limit. The ratio of $\sigma_{j=1/2}$ and the low-energy absorption cross section for S-wave massive scalar turns out to be dependent on the number of extra dimensions as $2^{-(n+3)/(n+1)}$. Thus, we can reproduce the ratio factor 1/8 at $n = 0$ which was shown by Unruh in ref. [6]. Of course, our result has a correct massless limit.

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For the case of the bulk scalar our result (4.23) for the low-energy absorption cross section is shown to have correct massless and $4d$ limits. Especially, the cross section for the massless S-wave is exactly same with the area of the horizon hypersurface, which is a higher-dimensional generalization of the universality for S-wave.

The extensions of our paper can be pursued in the several different directions. Firstly, one may consider the low-energy absorption cross section for a massive particle which has an arbitrary spin. In this case we should derive a radial master equation using the Newman-Penrose formalism. Then, it may be possible to examine the effect of spin in the absorption and emission problems of the higher-dimensional theories. Similar approach can be applied to generalize ref. [15]. Then we may understand the effect of particle's mass in the stringy description of black hole.

Since the matching method between the near-horizon and asymptotic solutions introduced in ref. [10, 11] gives an information on the high-energy absorption cross section, it may be possible to extend our paper to the extremely high-energy domain. It is of interest to check the effect of the extra dimensions in the high-energy absorption and emission problems.

Another extension of the present paper is to examine the absorption and emission problems in the entire range of energy. This can be performed numerically by following our recent paper [12]. In ref. [12] the absorption cross section for massive S-wave in the $4d$ black hole background exhibits a pecular behavior, i.e. decreasing behavior with increasing energy in the extremely low-energy regime. This behavior, as a result, breaks the universality for S-wave. It is interesting to check whether or not similar behavior exists in the higher-dimensional theories.

The most interesting one seems to be to understand the physical origin of the ratio factor $2^{-(n+3)/(n+1)}$. Since the absorption cross section for the brane-localized massless S-wave is fixed at $4\pi r_H^2$, this ratio factor indicates that the absorption cross section for the Dirac fermion increases when the extra dimensions exist. Especially, for the infinite extra dimensions the cross section for fermion becomes maximum, half of the cross section for scalar. Thus this ratio factor can be important to determine the number of the extra dimensions in the future collider.

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A. Comparision of $4d$ limit with $4d$ results

In this appendix we would like to show why the $n \to 0$ limit of our result (2.18) coincides with only the low-energy expansion of the usual $4d$ limit (1.4). As commented in section 2 the multiplicative and exponential factors in eq. (1.4) are results of the representation in terms of the Coulomb wave functions for an asymptotic solution in $4d$ case. We will show in this appendix that the expression of the asymptotic solution in terms of the Coulomb wave functions is impossible when extra dimensions exist.

For the proof it is convenient to rewrite eq. (2.5) as

$$\frac{d^2R}{dr^2} + \frac{1}{hr}\left[(n+1)(1-h)+2h\right]\frac{dR}{dr} + \left[\frac{\omega^2}{h^2} - \frac{1}{h}\left\{m^2 + \frac{\ell(\ell+1)}{r^2}\right\}\right]R = 0\,. \tag{A.1}$$

Using the fact that the first two terms in eq. (A.1) can be expressed as

$$\frac{1}{\sqrt{h}r}\frac{d^2}{dr^2}\left(\sqrt{h}rR\right) - (\text{no derivative terms})\,,$$

we can rewrite eq. (A.1) in the following form;

$$\left[\frac{d^2}{dr^2} + \left\{\frac{(n+1)(1-h)}{4r^2h^2}\left[(n-1)h+(n+1)\right] + \frac{\omega^2}{h^2} - \frac{1}{h}\left(m^2 + \frac{\ell(\ell+1)}{r^2}\right)\right\}\right](\sqrt{h}rR) = 0\,. \tag{A.2}$$

Now, let's taking $r \to \infty$ limit in eq. (A.2). Choosing the leading terms in the asymptotic region, we can show eq. (A.2) reduces to

$$\left[\frac{d^2}{dr^2} + \left\{(\omega^2 - m^2) + (2\omega^2 - m^2)\left(\frac{r_H}{r}\right)^{n+1} - \frac{\ell(\ell+1)}{r^2}\right\}\right](rR_{FF}) = 0\,. \tag{A.3}$$

For $n = 0$ eq. (A.3) yields a following asymptotic solution in terms of the Coulomb wave functions as expected

$$R_{FF}^{n=0} = A_I\frac{F_\ell^C\left(-\frac{(1+v^2)\omega r_H}{2v}, \omega v r\right)}{r} + A_{II}\frac{G_\ell^C\left(-\frac{(1+v^2)\omega r_H}{2v}, \omega v r\right)}{r}\,. \tag{A.4}$$

However, when extra dimensions exist, the asymptotic solution is expressed in terms of the Bessel function. Thus, the multiplicative factor and the exponential factor in the denominator of the Unruh's result (1.4) seem to be valid only when there is no extra dimension.

References

[1] S.A. Teukolsky, *Rotating black holes — separable wave equations for gravitational and electromagnetic perturbations*, *Phys. Rev. Lett.* **29** (1972) 1114.

[2] A.A. Starobinsky, *Amplification of waves during reflection from a rotating black hole*, *Zh. Eksp. Teor. Fiz.* **64** (1973) 48 [*Sov. Phys. JETP* **37** (1973) 28].

[3] B. Carter, *Charge and particle conservation in black hole decay*, *Phys. Rev. Lett.* **33** (1974) 558.

[4] L.H. Ford, *Quantization of a scalar field in the Kerr space-time*, *Phys. Rev.* **D 12** (1975) 2963.

[5] D.N. Page, *Particle emission rates from a black hole: massless particles from an uncharged, nonrotating hole*, *Phys. Rev.* **D 13** (1976) 198.

[6] W.G. Unruh, *Absorption cross-section of small black holes*, *Phys. Rev.* **D 14** (1976) 3251.

[7] N. Sanchez, *Absorption and emission spectra of a Schwarzschild black hole*, *Phys. Rev.* **D 18** (1978) 1030.

JHEP09(2004)005

[8] S.R. Das, G.W. Gibbons and S.D. Mathur, *Universality of low energy absorption cross sections for black holes*, *Phys. Rev. Lett.* **78** (1997) 417 [hep-th/9609052].

[9] R. Emparan, *Absorption of scalars by extended objects*, *Nucl. Phys.* **B 516** (1998) 297 [hep-th/9706204].

[10] D.K. Park and H.J.W. Müller-Kirsten, *Universality or non-universality of absorption cross sections for extended objects*, *Phys. Lett.* **B 492** (2000) 135 [hep-th/0008215].

[11] E. Jung, S.-H. Kim and D.K. Park, *Absorption cross section for s-wave massive scalar*, *Phys. Lett.* **B 586** (2004) 390 [hep-th/0311036].

[12] E. Jung and D.K. Park, *Effect of scalar mass in the absorption and emission spectra of Schwarzschild black hole*, *Class. and Quant. Grav.* **21** (2004) 3717 [hep-th/0403251].

[13] J.M. Maldacena, *Black holes and D-branes*, *Nucl. Phys.* **61A** *(Proc. Suppl.)* (1998) 111 [hep-th/9705078].

[14] J.M. Maldacena and A. Strominger, *Black hole greybody factors and D-brane spectroscopy*, *Phys. Rev.* **D 55** (1997) 861 [hep-th/9609026].

[15] M. Cvetič and F. Larsen, *Greybody factors for black holes in four dimensions: particles with spin*, *Phys. Rev.* **D 57** (1998) 6297 [hep-th/9712118].

[16] N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, *The hierarchy problem and new dimensions at a millimeter*, *Phys. Lett.* **B 429** (1998) 263 [hep-ph/9803315].

[17] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, *New dimensions at a millimeter to a Fermi and superstrings at a TeV*, *Phys. Lett.* **B 436** (1998) 257 [hep-ph/9804398].

[18] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370 [hep-ph/9905221].

[19] S.B. Giddings and S. Thomas, *High energy colliders as black hole factories: the end of short distance physics*, *Phys. Rev.* **D 65** (2002) 056010 [hep-ph/0106219].

[20] S. Dimopoulos and G. Landsberg, *Black holes at the LHC*, *Phys. Rev. Lett.* **87** (2001) 161602 [hep-ph/0106295].

[21] D.M. Eardley and S.B. Giddings, *Classical black hole production in high-energy collisions*, *Phys. Rev.* **D 66** (2002) 044011 [gr-qc/0201034].

[22] S.W. Hawking, *Breakdown of predictability in gravitational collapse*, *Phys. Rev.* **D 14** (1976) 2460.

[23] G.T. Horowitz and J. Maldacena, *The black hole final state*, *J. High Energy Phys.* **02** (2004) 008 [hep-th/0310281].

[24] P. Kanti and J. March-Russell, *Calculable corrections to brane black hole decay, I. The scalar case*, *Phys. Rev.* **D 66** (2002) 024023 [hep-ph/0203223].

[25] P. Kanti and J. March-Russell, *Calculable corrections to brane black hole decay, II. Greybody factors for spin 1/2 and 1*, *Phys. Rev.* **D 67** (2003) 104019 [hep-ph/0212199].

[26] C.M. Harris and P. Kanti, *Hawking radiation from a (4+n)-dimensional black hole: exact results for the Schwarzschild phase*, *J. High Energy Phys.* **10** (2003) 014 [hep-ph/0309054].

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[27] E. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, *J. Math. Phys.* **3** (1962) 566.

[28] R.C. Myers and M.J. Perry, *Black holes in higher dimensional space-times*, *Ann. Phys. (NY)* **172** (1986) 304.

[29] E. Schrödinger, *Commentat. Pontif. Acad. Sci.* **2** (1938) 321.

[30] S.S. Gubser, *Can the effective string see higher partial waves?*, *Phys. Rev.* **D 56** (1997) 4984 [hep-th/9704195].